Calculation of Transonic Flows Using an Extended Integral Equation Method

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Theme

THE extended integral equation method, previously applied[1] to transonic flows around thin nonlifting biconvex airfoils, is developed to treat the transonic flow around thick lifting round-nosed airfoils. In order to use the existing formulation of the transonic integral equation, the tangency boundary condition on the airfoil surface is replaced by an equivalent condition on the airfoil chord. The method is illustrated by examples both for shock-free flows and for flows with shock waves.

Content

At the present time most transonic flow calculations are performed using the purely numeric finite difference methods.[2] However, in recent years the semi-analytic integral equation method, solved approximately for nonlifting flows by several authors, has been extended to give an approximate theory for lifting subcritical flows[3] and latterly to remove the approximate element in the procedure to give the nominally exact extended integral equation method.[1] The basic aim of using an integral equation formulation is to try to perform analytically some of the operations performed numerically in the finite difference methods, and also to obtain from the analysis a useful insight into the character of the operations that are to be performed numerically, thus leading to reduced computing times.

The second-order differential equation

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{w}}{\partial \bar{z}}=\frac{\partial \bar{g}}{\partial \bar{x}}=\frac{\partial}{\partial \bar{x}}\left\{\frac{\bar{u}^2}{2}+\frac{\beta^2 M_\infty^2}{k}\bar{w}^2\right\} \tag{1}$$

can, with the equation of irrotationality, describe a steady two-dimensional transonic flow. If M_∞ is the freestream Mach number, $\beta=(1-M_\infty^2)^{1/2}$, and $k=k(\gamma, M_\infty)$ is a transonic parameter, where γ is the ratio of specific heats, then

$$\bar{u}(\bar{x},\bar{z})=\frac{k}{\beta^2}u(x,z) \qquad \bar{w}(\bar{x},\bar{z})=\frac{k}{\beta^3}w(x,z) \qquad \begin{pmatrix}\bar{x}=x\\ \bar{z}=\beta z\end{pmatrix} \tag{2}$$

where $u(x,z)$, $w(x,z)$ are the nondimensional perturbation velocities in the x and z directions, respectively, and (x,z) is a Cartesian coordinate system with the origin at the leading edge of the airfoil. Equation (1), together with the irrotationality condition, is to be solved subject to the usual boundary conditions for transonic potential flows.

The usual formulation of the transonic integral equation requires that the tangency boundary condition be satisfied on the plane $\bar{z}=\pm 0$. In order to treat the flow over thick lifting airfoils without a fundamental change in the formulation of the integral equation, it is assumed that the real external flow can be analytically continued inside the airfoil to give the equivalent boundary condition on the plane $\bar{z}=\pm 0$. This is obtained by taking the first two terms of a Taylors series expansion which, for Eq. (1) and its associated boundary conditions, gives

$$\bar{w}(\bar{x},+0)=\frac{k}{\beta^3}\left\{\frac{\bar{z}_{ux}(\bar{x})}{\beta}\left[\cos(\alpha)+\frac{\beta^2}{k}\bar{u}(\bar{x},\bar{z}_u)\right]-\sin(\alpha)\right\}$$
$$+\bar{z}_u(\bar{x})\left[\bar{u}_{\bar{x}}(\bar{x},\bar{z}_u)-\bar{g}_{\bar{x}}(\bar{x},\bar{z}_u)\right]$$
$$\bar{w}(\bar{x},-0)=\frac{k}{\beta^3}\left\{\frac{\bar{z}_{l_x}(\bar{x})}{\beta}\left[\cos(\alpha)+\frac{\beta^2}{k}\bar{u}(\bar{x},\bar{z}_l)\right]-\sin(\alpha)\right\}$$
$$+\bar{z}_l(\bar{x})\left[\bar{u}_{\bar{x}}(\bar{x},\bar{z}_l)-g_{\bar{x}}(\bar{x},\bar{z}_l)\right] \tag{3}$$

where α is the angle of incidence and $\bar{z}=\bar{z}_u(\bar{x})$, $\bar{z}=\bar{z}_l(\bar{x})$ describe the upper and lower surfaces of the airfoil, respectively, in the transformed coordinates $(\bar{x},\bar{z})$.

The differential equation, Eq. (1), with the irrotationality condition and subject to the given boundary conditions, can be inverted into integral form using Green's theorem to give

$$\bar{u}(\bar{x},\bar{z})-\bar{g}(\bar{x},\bar{z})=I_l(\bar{x},\bar{z})+I_s(\bar{x},\bar{z},\bar{x}_s) \tag{4}$$

where $\bar{x}=\bar{x}_s(\bar{z})$ denotes the shock location. For $\bar{z}\neq 0$

$$I_l(\bar{x},\bar{z})=\frac{1}{2\pi}\int_0^1\left[\psi_{\bar{x}}(r)\right]_{\zeta=0}\Delta\bar{w}(\xi)\mathrm{d}\xi$$
$$-\frac{1}{2\pi}\int_0^1\left[\psi_{\bar{z}}(r)\right]_{\zeta=0}\Delta\bar{u}(\xi)\mathrm{d}\xi \tag{5a}$$

$$I_s(\bar{x},\bar{z},\bar{x}_s)=-\frac{1}{2\pi}\int_S\int\psi_{\xi\bar{x}}(r)\bar{g}(\xi,\zeta)\mathrm{d}\xi\mathrm{d}\zeta \tag{6a}$$

and for $\bar{z}=\pm 0$)

$$I_l(\bar{x}\pm 0)=\frac{1}{2\pi}\int_0^1\frac{\Delta\bar{w}(\xi)}{(\bar{x}-\xi)}\mathrm{d}\xi$$
$$+\frac{1}{2\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{1/2}\int_0^1\frac{\bar{w}_T(\xi)}{(\bar{x}-\xi)}\left(\frac{\xi}{1-\xi}\right)^{1/2}\mathrm{d}\xi$$

$$I_s(\bar{x},\pm 0,\bar{x}_s)=\frac{1}{4\pi}\int_S\int\left[\psi_{\xi_{\bar{x}}}(r)\right]_{\bar{z}=0}$$
$$\times\left[\bar{g}(\xi-\zeta)+\bar{g}(\xi,-\zeta)\right]\mathrm{d}\xi\mathrm{d}\zeta$$
$$\mp\frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{1/2}\int_0^1\frac{I_c(\xi,\bar{x}_s)}{(\bar{x}-\xi)}\left(\frac{\xi}{1-\xi}\right)^{1/2}\mathrm{d}\xi \tag{6b}$$

The kernel function ψ is given by

$$\psi(r)=\ln|r|=\tfrac{1}{2}\ln\left[(\bar{x}-\xi)^2)+(\bar{z}-\zeta)^2\right]$$

the operator Δ is defined for a function $f(\bar{x},\bar{z})$ by

$$\Delta f(\bar{x})=f(\bar{x},+0)-f(\bar{x},-0)$$

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and

$$\bar{w}_T(\bar{x}) = \bar{w}(\bar{x}, +0) + \bar{w}(\bar{x}, -0)$$

The integral $I_c(\bar{x},\bar{x}_s)$ is defined by

$$I_c(\bar{x},\bar{x}_s) = -\frac{1}{2\pi}\int_S\int\Big[\psi_{\xi z}(r)\Big]_{\bar{z}=+0}\Big[\bar{g}(\bar{\xi},\bar{\zeta}) - \bar{g}(\bar{\xi},\pm 0)\Big]\,d\bar{\xi}\,d\bar{\zeta}$$

where

$$\bar{g}(\bar{\xi},\pm 0) = \begin{cases}\bar{g}(\bar{\xi},+0), & \bar{\zeta}>0\\ \bar{g}(\bar{\xi},-0), & \bar{\zeta}<0\end{cases}$$

The infinite domain S contains the shock location $\bar{x}_s(\bar{z})$ implicitly. The Kutta condition is satisfied implicitly in the inversion leading to Eqs. (5b, 6b). Equation (4), with Eqs. (5, 6), is the extended transonic integral equation; the much simpler standard integral equation[3] is given by Eq. (4) with Eqs. (5b, 6b).

Equation (6) relates the velocity components $\bar{u}(\bar{x},\bar{z})$, $\bar{w}(\bar{x},\bar{z})$; the necessary second equation is obtained from the irrotationality condition. As shock waves are present in the flow, then in order to ensure finite acceleration everywhere except at the shock wave, the solution must satisfy the following regularity conditions, which give the shock location

$$\left\{\frac{\beta^2 M_\infty^2}{k}\bar{w}^2(\bar{x},\bar{z}) + I_I(\bar{x},\bar{z}) + I_s(\bar{x},\bar{z},\bar{x}_s)\right\}_{\bar{x}=\bar{x}_0(\bar{z})} = 1/2$$

$$\frac{\partial}{\partial\bar{x}}\left\{\frac{\beta^2 M_\infty^2}{k}\bar{w}^2(\bar{x},\bar{z}) + I_I(\bar{x},\bar{z}) + I_s(\bar{x},\bar{z},\bar{x}_s)\right\}\Big|_{\bar{x}=\bar{x}_0(\bar{z})} = 0 \qquad (7)$$

where $\bar{x}_0(\bar{z})$ is defined by $\bar{u}(\bar{x}_0(\bar{z}),\bar{z}) = 1$.

The field integrals in Eqs. (4) are evaluated by dividing the flowfield into a finite number of strips parallel to the $\bar{x}$-axis and approximating $\bar{g}(\bar{\xi},\bar{\zeta})$ in each strip by straight-line interpolation in terms of values on the strip edges. The details are given in the source paper.

The pressure distributions for three test cases are shown in Figs. 1-3. In the subcritical example shown in Fig. 1,

$$k = (3 + (\gamma - 2)M_\infty^2)M_\infty^2$$

In the remaining examples with shock waves the function $\bar{g}(\bar{x},\bar{z})$ is defined as

$$\bar{g}(\bar{x},\bar{z}) = \bar{u}^2(\bar{x},\bar{z})/2$$

with $k = (\gamma + 1)M_\infty^{3/2}$

For the subcritical example, shown in Fig. 1, the present method gives fairly good agreement with the accurate result of Sells[4] and is a considerable improvement on the results of the "standard" integral equation method.[3] In Fig. 2 the results of the present method are compared with the "nonconservative" result of the Garabedian-Korn method. In Fig. 3 the results of the present method are compared to the results of the conservative and "nonconservative" finite difference methods.[2] The present results agree fairly well with the incorrect "nonconservative" result rather than the correct conservative result. It is thought that this is due to the use of a much more simplifed form of the potential equation than that used in the finite difference results. The computing times for the present method are about half those of an equivalent relaxation finite difference solution.

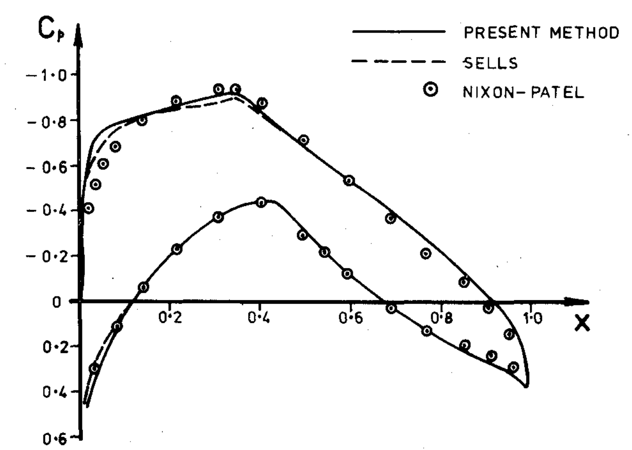

Fig. 1 Pressure distribution around an NPL 3111 airfoil; $M_\infty = 0.667$, $\alpha = 1.2°$.

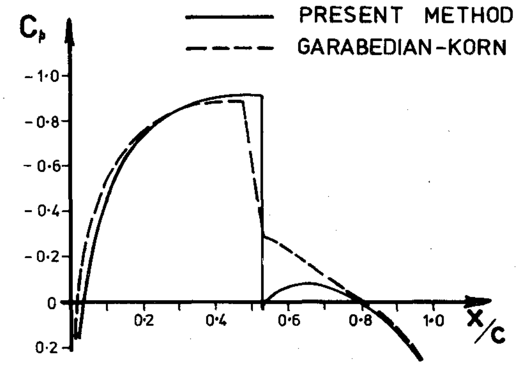

Fig. 2 Pressure distribution around an NACA 0012 airfoil; $M_\infty = 0.816$, $\alpha = 0.0°$.

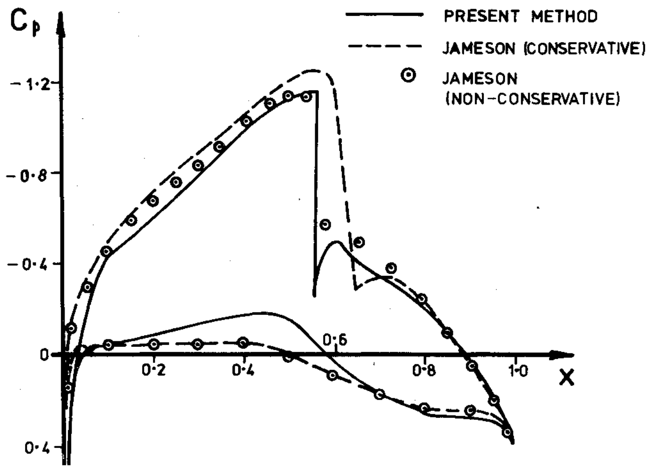

Fig. 3 Pressure distribution around an NACA 64A410 airfoil; $M_\infty = 0.72$, $\alpha = 0°$.

It is possible that the computing time could be reduced further by incorporating more analytic steps into the theory in addition to improving the computational efficiency. Any further work in steady two-dimensional transonic flows should be directed to this problem, together with the addition of more terms into the potential equation and the provision for fitting a curved shock wave.

As stated earlier, the main aim of the integral equation method is to use analysis to replace some numerical operations, and any further extensions to the theory are worthwhile only if a significant saving in computing time is possible. For example, such a saving is likely if the theory is extended to oscillatory flows; it is not easily ascertainable if an extension to steady three-dimensional flows would result in radically faster computing times than existing finite difference solutions.

References

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